

Stationary distribution functions for Tokamak-plasmas in the weak-collisional transport regime by MaxEnt principle

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In previous works, we derived stationary density distribution functions (DDF) where the local equilibrium is determined by imposing the maximum entropy (MaxEnt) principle, under the scale invariance restrictions, and the minimum entropy production theorem. In this paper we demonstrate that it is possible to reobtain these DDF solely from the MaxEnt principle subject to suitable scale invariant restrictions in all the variables. For the sake of concreteness, we analyze the example of ohmic, fully ionized, tokamak-plasmas, in the weak-collisional transport regime. In this case we show that it is possible to reinterpret the stationary distribution function in terms of the Prigogine distribution function where the logarithm of the DDF is directly linked to the entropy production of the plasma. This leads to the suggestive idea that also the stationary neoclassical distribution functions, for magnetically confined plasmas in the collisional transport regimes, may be derived solely by the MaxEnt principle.

1. Introduction

In recent works we determined the expression of a stationary density distribution function (DDF), previously proposed in literature, by statistical thermodynamics (Sonnino *et al.* 2013)-(Sonnino *et al.* 2014). This DDF is denoted by $\mathcal{F}^{\alpha R}$ (with $\alpha = e, i$ for electrons and ions, respectively). The stationary DDF has been determined in three steps. Firstly, we considered open thermodynamic systems close to a local equilibrium state obeying to Prigogine's statistical thermodynamics. Successively, we defined the local equilibrium state by adopting a minimal number of hypotheses : the minimum entropy production (MEP) theorem and the MaxEnt principle, under two scale invariance restrictions. Finally, we linked the Prigogine probability distribution function with the DDF, which is obtained by perturbing the local equilibrium state (LES).

The gyrokinetic (GK) theory makes often use of an initial distribution function of guiding centers. In the GK simulations, as well as in the GK theory, this initial distribution

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function is usually taken as a reference density distribution function if it depends only on the invariants of motion and it evolves slowly from the local equilibrium state i.e., in such a way that the guiding centers remain confined for sufficiently long time.

The aim of this work is to show that the DDF may be derived by using solely the MaxEnt principle, subject to suitable restrictions. For the sake of concreteness, we shall analyze Tokamak ohmic-plasmas in the weak-collisional transport regime. We shall see that, in this case, the obtained DDF may be re-interpreted in terms of Prigogine's distribution function where the entropy production of the plasma is proportional to the logarithm of the DDF.

The paper is organized as follows. In Section 2, we shall re-derive, very briefly, the DDF derived by non-equilibrium statistical mechanics, by using *solely* the MaxEnt principle, under suitable scale invariance restrictions, without using the MEP. Of course, from the physical point of view, the special choice of the scale invariance restrictions can be justified by evoking the MEP. Section 3 treats tokamak-plasmas in the weak collisional transport regime. We shall see that it is possible to set the free parameters appearing in the DDF in such a way that the logarithm of this stationary distribution function coincides exactly with the entropy production of the plasma estimated by the neoclassical theory. This leads to the suggestive idea that also the stationary distribution function derived by neoclassical theory may be obtained by MaxEnt principle (under suitable restrictions).

2. Derivation of a Stationary Distribution Function, Recently Mentioned in Literature, by MaxEnt Principle

In this Section we shall prove that the DDF derived by non-equilibrium statistical mechanics (Sonnino *et al.* 2013)-(Sonnino *et al.* 2014) can easily be obtained from maximum entropy principle with suitable scale invariant restrictions in all the variables. In the case of an axisymmetric magnetically confined plasma, after having performed the guiding center transformation, the necessary variables for describing the system reduce to four independent variables (Balescu, Vol. 1 1988). These variables are defined as follows. One of these ones is the *poloidal magnetic flux*, ψ , and another variable is the *particle kinetic energy per unit mass*, w , defined as $w = (v_{\parallel}^2 + v_{\perp}^2)/2$ with v_{\parallel} denoting the parallel component of particle's velocity (which may actually be parallel or antiparallel to the magnetic field), and v_{\perp} the absolute value of the perpendicular velocity (Balescu, Vol. 1 1988). The remaining two variables are the *toroidal angular momentum* P_{ϕ} and the variable linked to the pitch angle, λ . These quantities are defined as (for a rigorous definition, see any standard textbook such as, for example, (Balescu Vol 2 1988))

$$P_{\phi} = \psi + \frac{B_0}{\Omega_{0c}} \frac{F v_{\parallel}}{|B|} \quad (2.1)$$

$$\lambda \equiv \frac{\mu}{w} = \frac{\sin^2 \theta_P}{2 |B|} \quad \text{with} \quad \mu = \frac{v_{\perp}^2}{2 |B|} \quad (2.2)$$

Here Ω_{0c} is the *cyclotron frequency* associated with the magnetic field along the magnetic axis, B_0 . $|B|$, F and θ_P denote the *magnetic field intensity*, the *characteristic of axisymmetric toroidal field* depending on the *poloidal magnetic flux* ψ and the *pitch angle*, respectively.

The expression for the reference (density of) distribution function \mathcal{F}^R reads†

$$\mathcal{F}^R = \mathcal{N}_0 \left[\frac{w}{\Theta} \right]^{\gamma-1} \exp[-w/\Theta] \exp \left[- \left(\frac{P_\phi - P_{\phi 0}}{\Delta P_\phi} \right)^2 \right] \exp \left[- \left(\frac{\Delta \lambda_0}{\Delta \lambda_1} + \frac{w}{\Theta} \right) \frac{(\lambda - \lambda_0)^2}{\Delta \lambda_0} \right] \quad (2.3)$$

where \mathcal{N}_0 ensures normalization to unity in the reduced phase space $\hat{\Omega}$. Parameters γ , ΔP_ϕ , $\Delta \lambda_0$, $\Delta \lambda_1$, $P_{\phi 0}$ and λ_0 are, at this stage, six free parameters. However, it is correct to mention that Eq. (2.3) is able to describe only a quite limited number of physical scenarios. Indeed, as it is shown in Refs (Sonnino *et al.* 2013)-(Sonnino *et al.* 2014), Eq. (2.3) correspond to stationary DDFs, written at the lowest order, obtained by adopting a very special choice of local equilibrium. In particular, this expression is inadequate for treating dynamical systems in highly anisotropic phase space. In Ref. (Sonnino and Steinbrecher 2014) we show that a large class of anisotropic dynamical systems subject to random perturbations, including particle transport in random media, can adequately be treated by assuming the validity of the MaxEnt principle of the generalized Rényi entropies. In these cases, these entropies play the role of Liapunov functionals (Sonnino and Steinbrecher 2014).

Let us proof now that the DDF in Eq (2.3) can easily be derived by applying the MaxEnt principle, subject to some restrictions, to the Shannon entropy.

The reduced phase space is defined by: $w \geq 0$, $\lambda \geq 0$ and $P_\phi \in \mathbb{R}$. It is invariant under the scaling of the variables

$$w \rightarrow aw \quad , \quad \lambda \rightarrow b\lambda \quad , \quad P_\phi \rightarrow sP_\phi \quad (2.4)$$

where $a, b, s > 0$ are arbitrary parameters of the dilatations group acting in $\hat{\Omega}$. We use the normalization

$$\int_{\hat{\Omega}} \mathcal{F}^R d\Gamma = 1 \quad \text{where} \quad d\Gamma = |\mathcal{J}| dw dP_\phi d\lambda \quad (2.5)$$

with \mathcal{J} denoting the Jacobian. Since the DDF is a pseudo scalar, we have the following invariance property under the change of variables Eq. (2.4)

$$\mathcal{F}^R d\Gamma \rightarrow \mathcal{F}^R d\Gamma \quad (2.6)$$

The Boltzmann-Shannon entropy (Shannon 1948) $\mathcal{S}(f)$ corresponding to an arbitrary probability distribution function (PDF) $f(w, P_\phi, \lambda)$ in this phase space $\hat{\Omega}$ is

$$\mathcal{S}(f) = - \int_{\hat{\Omega}} f \log f d\Gamma \quad (2.7)$$

We can recover the form Eq. (2.3) of the DDF by using the Gibbs theorem:

The DDF corresponding to maximum of the entropy (2.7) with linear restrictions gener-

† We mention that this DDF has been previously proposed in Ref. (Di Troia 2012) with $\Delta \lambda_1 \rightarrow \infty$. The DDF in Ref. (Di Troia 2012) has been introduced on the base of the Ion Cyclotron Radiation Heating (ICRH) FAST-plasma, simulated by using the Hybrid Magneto-hydrodynamic Gyro-kinetic Code (HMGC) (Cardinali *et al.* 2010) and (Pizzuto *et al.* 2010). However, the distribution function in Ref. (Di Troia 2012) should be understood only as an *ad hoc* expression, which has been introduced for fitting the steady state equilibria in several physical scenarios by five free parameters devoid, at that level, of any physical interpretation. Hence, apart the presence of parameter $\Delta \lambda_1$, the major difference between the works in (Sonnino *et al.* 2013)-(Sonnino *et al.* 2014) and the paper in Ref. (Di Troia 2012) stands in the derivation method; based in the former works on a rigorous statistical mechanics approach.

ated by a set of phase space functions $g_0(w, P_\phi, \lambda), \dots, g_N(w, P_\phi, \lambda)$ of the form

$$\int_{\hat{\Omega}} f g_k d\Gamma = c_k; \quad 0 \leq k \leq N \quad (2.8)$$

is given by

$$f(w, P_\phi, \lambda) = \exp \left[- \sum_{k=0}^N \mu_k g_k(w, P_\phi, \lambda) \right] \quad (2.9)$$

where μ_k are the Lagrange multipliers that are fixed from the constraints Eq. (2.8).

Remark that the normalization Eq. (2.9) is assured by setting

$$g_0(w, P_\phi, \lambda) \equiv 1 \quad (2.10)$$

In analogy to the approach used in the previous work (Sonnino *et al.* 2013), (Sonnino *et al.* 2014), where the dependence of the DDF Eq. (2.3) of the single variable w was explained by using the maximal entropy principle with scale invariant restrictions, we consider, in addition to the restriction related to the normalization condition Eq. (2.10) the following scale invariant restrictions, previously used in (Sonnino *et al.* 2013), (Sonnino *et al.* 2014)

$$g_1(w, P_\phi, \lambda) \equiv \log(w) \quad (2.11)$$

$$g_2(w, P_\phi, \lambda) \equiv w \quad (2.12)$$

In order to derive the Gaussian dependence of the variable P_ϕ , we use the well known homogenous functions

$$g_3(w, P_\phi, \lambda) \equiv P_\phi \quad (2.13)$$

$$g_4(w, P_\phi, \lambda) \equiv P_\phi^2 \quad (2.14)$$

The λ dependence of the density distribution function is assured by the following monomials

$$g_5(w, P_\phi, \lambda) \equiv \lambda \quad (2.15)$$

$$g_6(w, P_\phi, \lambda) \equiv \lambda^2 \quad (2.16)$$

$$g_7(w, P_\phi, \lambda) \equiv w\lambda \quad (2.17)$$

$$g_8(w, P_\phi, \lambda) \equiv w\lambda^2 \quad (2.18)$$

As shown in Ref. (Sonnino *et al.* 2014), from the physical point of view, the *ad hoc* restrictions Eqs (2.13-2.18), may be justified *a posteriori* by evoking the minimum entropy production theorem. According to Eq. (2.6) and the previous form of functions $g_k(w, P_\phi, \lambda)$ from Eqs (2.10-2.18), the functional form of the linear restrictions Eq. (2.8) remains unchanged, only the coefficients c_1, \dots, c_8 are changed

$$c_1 \rightarrow c_1 + \log a \quad (2.19)$$

$$c_2 \rightarrow ac_2 \quad (2.20)$$

$$c_3 \rightarrow sc_3 \quad (2.21)$$

$$c_4 \rightarrow s^2c_4 \quad (2.22)$$

$$c_5 \rightarrow bc_5 \quad (2.23)$$

$$c_6 \rightarrow b^2c_6 \quad (2.24)$$

$$c_7 \rightarrow abc_7 \quad (2.25)$$

$$c_8 \rightarrow ab^2c_8 \quad (2.26)$$

From the previous formulae it follows that the linear affine sub-manifold \mathcal{L} in the functional space of probability density functions, defined by the linear restrictions Eqs (2.8, 2.10-2.18), after the rescaling of the variables, is translated by a constant vector. We can recover the DDF from Eq. (2.3) by using Eq. (2.9), with the functions $g_{0,\dots,9}(w, P_\phi, \lambda)$ given by Eqs (2.10-2.18) and the following choice of the Lagrange multipliers

$$\mu_1 = \gamma - 1 \quad (2.27)$$

$$\mu_2 = \frac{1}{\Theta} \left[1 + \left(\frac{\lambda_0}{\Delta\lambda_0} \right)^2 \right] \quad (2.28)$$

$$\mu_3 = -\frac{2P_{\phi 0}}{\Delta P_\phi^2} \quad ; \quad \mu_4 = \frac{1}{\Delta P_\phi^2} \quad (2.29)$$

$$\mu_5 = \frac{-2\lambda_0}{\Delta\lambda_1 \Delta\lambda_0} \quad ; \quad \mu_6 = \frac{1}{\Delta\lambda_1 \Delta\lambda_0} \quad (2.30)$$

$$\mu_7 = \frac{-2\lambda_0}{\Theta \Delta\lambda_0^2} \quad ; \quad \mu_8 = \frac{1}{\Theta \Delta\lambda_0^2} \quad (2.31)$$

Note that the presence of the free parameter $\Delta\lambda_1$ in Eq. (2.3) is crucial. Indeed, as we shall show in the next Section, the absence of $\Delta\lambda_1$ precludes the possibility of identifying the DDF, given by Eq. (2.3), with the one estimated by the neoclassical theory for collisional tokamak-plasmas (see, for example, Ref. (Balescu Vol 2 1988)). In addition, it allows describing more complex physical scenarios such as, for example, the *modified bi-Maxwellian* distribution function. Last and not least, in some physical circumstances, the presence of $\Delta\lambda_1$ is essential to ensure the normalization of the DDF.

3. Estimation of the Free Parameters in the DDF for, Ohmic, Fully Ionized, Tokamak-Plasmas, in the weak-collisional Transport Regime

In this section, we show that it is possible to set up the free parameters appearing in the DDF in Eq. (2.3) in such a way that the logarithm of the DDF coincides exactly with entropy production of tokamak-plasmas in the weak-collisional transport regime. This means that, ultimately, the DDF may be re-interpreted in terms of the Prigogine theory on the distribution functions for fluctuation of a thermodynamic variable. This leads to the attractive idea that the stationary distribution function, derived by the neoclassical theory, may also be obtained by the MaxEnt principle subject to suitable restrictions. The density probability distribution of finding a state in which the values of the fluctuation of a thermodynamic variable, $\tilde{\alpha}_i$, lies between $\tilde{\alpha}_i$ and $\tilde{\alpha}_i + d\tilde{\alpha}_i$ is (Boltzmann's constant is set to 1) (Prigogine 1947), (Prigogine 1954)

$$\mathcal{F} = \mathcal{N}_0 \exp[-\Delta_I S^\alpha] \quad (\alpha = e, i) \quad (3.1)$$

where \mathcal{N}_0 ensures normalization to unity. The entropy production $\Delta_I S$ (≥ 0) is linked to the entropy production strength σ , the thermodynamic forces X^μ , and the thermodynamic flow J_μ by (De Groot and Mazur 1984)

$$\Delta_I S^\alpha = \int_\Omega \sigma^\alpha d\mathbf{x} dt = \int_\Omega \frac{n_\alpha}{\tau_\alpha} \Delta_I \tilde{S}^\alpha d\mathbf{x} dt = \int_\Omega X^\mu J_\mu d\mathbf{x} dt \geq 0 \quad (3.2)$$

Here, n_α , τ_α and $\Delta_I \tilde{S}^\alpha$ denote the number density, the relaxation time and the dimensionless entropy production (per unit particle) of species α , respectively. $d\mathbf{x}$ is the spatial

volume element and, in the case of Tokamak-plasmas, the integration is made over an annular shell, Ω , contained between two adjacent magnetic surfaces $\psi, \psi + d\psi$. Integrating over time of the order of the collision time, we get $\Delta_I \hat{S} \simeq \Delta_I \tilde{S}$, with $\Delta_I \hat{S}$ denoting the entropy production per unit particle. Note that the probability density function (3.2) remains unaltered for flux-force transformations, $X_\kappa \rightarrow X'_\kappa$ and $J_\kappa \rightarrow J'_\kappa$, leaving invariant the entropy production. This property is referred to as the *Thermodynamic Covariance Principle* (TCP) (Sonnino 2009), (G. Sonnino and A. Sonnino 2014). Hence, according to Prigogine's formalism, from Eqs (3.1) and (3.2), we see that two density distribution functions coincide if, and only if, the two expressions of $\Delta_I \tilde{S}^\alpha$ are identical *for all values taken by the variables*.

For easy reference, we report the main balance equations linking the DDF with entropy (i.e., the entropy production strength and the flux entropy).

• **The equation for the entropy production strength**

$$\sigma^\alpha = \frac{n_\alpha}{\tau_\alpha} \Delta_I \tilde{S}^\alpha = - \sum_{\beta=e,i} \int_{\mathcal{V}} d\mathbf{v} [\ln \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x})] \mathcal{K}^{\alpha\beta} \quad (3.3)$$

with $\mathcal{K}^{\alpha\beta}$ denoting the collisional operator of species α due to β and \mathcal{V} the velocity-volume in the phase-space, respectively.

• **The equation for the flux entropy**

$$\mathbf{J}_S^\alpha(\mathbf{x}) = - \int_{\mathcal{V}} d\mathbf{v} [\mathbf{v} - \mathbf{u}_\alpha(\mathbf{x}, \tilde{\Theta}_\alpha)] \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x}) \ln \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x}, \tilde{\Theta}_\alpha) = \frac{\mathbf{J}_\mathcal{E}}{T_\alpha} \quad (3.4)$$

with $\mathbf{J}_\mathcal{E}$ denoting the total energy flux, and \mathbf{u}^α and T_α the mean velocity and temperature, respectively. The number density, the mean velocity and temperature, are provided by

$$n_\alpha(\mathbf{x}) = \int_{\mathcal{V}} d\mathbf{v} \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x}) \quad ; \quad n_\alpha(\mathbf{x}) \mathbf{u}^\alpha(\mathbf{x}) = \int_{\mathcal{V}} d\mathbf{v} \mathbf{v} \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x}) \quad (3.5)$$

and

$$n_\alpha(\mathbf{x}) T_\alpha(\mathbf{x}) = \frac{1}{2} m_\alpha \int_{\mathcal{V}} d\mathbf{v} |\mathbf{v} - \mathbf{u}_\alpha|^2 \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x}) \quad (3.6)$$

m_α is the mass particle of specie α .

In this work we consider ohmic, fully ionized, tokamak-plasmas, defined as a collection of magnetically confined electrons and positively charged ions. According to the neo-classical estimation, the dimensionless entropy production of species α , $\Delta_I \tilde{S}^\alpha$, is derived under the sole assumption that the state of the quiescent plasma is not too far from the reference local Maxwellian. In the *local dynamical triad*, and up to the second order of the drift parameter, the dimensionless entropy production can be brought into the form (see Refs (Balescu Vol 2 1988) and (Sonnino and Peeters 2008))

$$\begin{aligned} \Delta_I \tilde{S}_{neocl}^e &= \tilde{\sigma}_\parallel (g_\parallel^{(1)} - \bar{g}_\parallel^{e(1)})^2 + \tilde{\kappa}_\parallel^e (g_\parallel^{e(3)} + \bar{g}_\parallel^{e(3)})^2 + \tilde{\epsilon}_\parallel^e (\bar{g}_\parallel^{e(5)})^2 \\ &\quad + 2\tilde{\alpha}_\parallel (g_\parallel^{(1)} - \bar{g}_\parallel^{e(1)}) (g_\parallel^{e(3)} + \bar{g}_\parallel^{e(3)}) + 2\tilde{\gamma}_\parallel (g_\parallel^{(1)} - \bar{g}_\parallel^{e(1)}) \bar{g}_\parallel^{e(5)} + 2\tilde{\delta}_\parallel^e (g_\parallel^{e(3)} \\ &\quad + \bar{g}_\parallel^{e(3)}) \bar{g}_\parallel^{e(5)} + \tilde{\sigma}_\perp (g_\rho^{(1)P})^2 + \tilde{\kappa}_\perp^e (g_\rho^{e(3)})^2 - 2\tilde{\alpha}_\perp g_\rho^{(1)P} g_\rho^{e(3)} \\ \Delta_I \tilde{S}_{neocl}^i &= \tilde{\kappa}_\parallel^i (g_\parallel^{i(3)} + \bar{g}_\parallel^{i(3)})^2 + \tilde{\epsilon}_\parallel^i (\bar{g}_\parallel^{i(5)})^2 + 2\tilde{\delta}_\parallel^i (g_\parallel^{i(3)} + \bar{g}_\parallel^{i(3)}) \bar{g}_\parallel^{i(5)} + \tilde{\kappa}_\perp^i (g_\rho^{i(3)})^2 \end{aligned} \quad (3.7)$$

Here $q_r^{\alpha(n)}$ [with $r = (\rho, \parallel, \wedge)$] denote the *Hermitian moments of the distribution functions* and $g_r^{\alpha(n)}, \bar{g}_r^{\alpha(n)}$ are the *dimensionless source terms*. In particular, $g_\parallel^{(1)}$ and $g_\parallel^{\alpha(3)}$ are dimensionless parallel component of the modified electric field and of the temperature gradient of species α , respectively. $\bar{g}_\parallel^{\alpha(n)}$ denote the parallel components of the

additional sources, in the long mean-free-path regime (their expressions can be found in Ref. (Balescu Vol 2 1988)). Coefficients $\tilde{\sigma}_r$, $\tilde{\alpha}_r$, $\tilde{\kappa}_r^\alpha$ are the dimensionless component of the *electronic conductivity*, the *thermoelectric coefficient* and the *electric* ($\alpha = e$) or *ion* ($\alpha = i$) *thermal conductivity*, respectively. Moreover, $\tilde{\gamma}_\parallel$, $\tilde{\delta}_\parallel^\alpha$ and $\tilde{\epsilon}_\parallel^\alpha$ are the *parallel transport coefficients* in 21M approximation.

In the previous section and in Refs (Sonnino *et al.* 2013), (Sonnino *et al.* 2014), we have shown that, by information theory (MaxEnt principle) or by statistical thermodynamics, it is possible to determine the expression of the DDF, without being able, however, to fix the values of the seven parameters ΔP_ϕ , $\Delta\lambda_0$, $\Delta\lambda_1$ and Θ . We shall see that the first three coefficients are linked to the transport coefficient whereas Θ to the source.

Ultimately, Eq. (2.3) represents the reference distribution function for a test particle. Hence, the logarithm of this DDF may be re-interpreted as the entropy production per unit particle (i.e., as $\Delta_I \hat{S}$). In Ref. (Sonnino *et al.* 2014) it is shown that the logarithm of the DDF, Eq. (2.3), can be brought into the form

$$\Delta_I \hat{S} = -(\gamma - 1) \ln\left(\frac{w}{\Theta}\right) + \frac{w}{\Theta} + \frac{1}{2} g_{11} \tilde{\alpha}_1^2 + \frac{1}{2} g_{22} \tilde{\alpha}_2^2 + g_{12} \tilde{\alpha}_1 \tilde{\alpha}_2 \quad (3.8)$$

where fluctuations $\tilde{\alpha}_\kappa$ ($\kappa = 1, 2$) are linked to the thermodynamic forces X^κ and flows J_κ , by the relations (Onsager, Vol 37 1931), (Onsager, Vol 38 1931)

$$X^\kappa = \frac{\partial \Delta_I \hat{S}}{\partial \tilde{\alpha}_\kappa} \quad ; \quad J_\kappa = \frac{d \tilde{\alpha}_\kappa}{dt} \quad (3.9)$$

By expanding the previous expression around the reference value $w = w_0$ we obtain, up to the second order

$$\begin{aligned} \Delta_I \hat{S} = & \left[-(\gamma - 1) \ln\left(\frac{w_0}{\Theta}\right) + \frac{w_0}{\Theta} \right] + \left[-(\gamma - 1) \frac{1}{w_0} + \frac{1}{\Theta} \right] (w - w_0) \\ & + (\gamma - 1) \frac{1}{2w_0^2} (w - w_0)^2 + \frac{1}{2} g_{11} \tilde{\alpha}_1^2 + \frac{1}{2} g_{22} \tilde{\alpha}_2^2 + g_{12} \tilde{\alpha}_1 \tilde{\alpha}_2 + h.o.t. \end{aligned} \quad (3.10)$$

where *h.o.t.* stands for higher order terms. Up to a normalization constant, we have that the distribution function Eq. (2.3) is approximated by a Gaussian density distribution function (in the variable w) by setting to zero the coefficient of the linear term [i.e., $-(\gamma - 1)/w_0 + 1/\Theta = 0$]. The *global optimality conditions* is obtained by imposing that also the sum of the constant terms vanishes [i.e., $-(\gamma - 1) \ln(w_0/\Theta) + w_0/\Theta = 0$]. These two requirements are simultaneously satisfied only if

$$\gamma = 1 + E \quad ; \quad w_0 = (\gamma - 1)\Theta = E\Theta \quad (3.11)$$

Hence, solutions (3.11) ensure not only a local approximation, valid up to the second order, but also a good global approximation. In this sense the values of w_0 and γ , provided by Eq. (3.11), are *optimal*. From Eqs (3.9) and (3.10) we obtain the expressions of the thermodynamic forces

$$\begin{aligned} X^1 & \equiv \frac{\partial \Delta_I \hat{S}}{\partial \tilde{\alpha}_1} = g_{11} \tilde{\alpha}_1 + g_{12} \tilde{\alpha}_2 \\ X^2 & \equiv \frac{\partial \Delta_I \hat{S}}{\partial \tilde{\alpha}_2} = g_{12} \tilde{\alpha}_1 + g_{22} \tilde{\alpha}_2 \\ X^3 & \equiv \frac{\partial \Delta_I \hat{S}}{\partial w} = \frac{1}{E\Theta^2} (w - w_0) \end{aligned} \quad (3.12)$$

By solving the previous system of equations with respect to α_1 , α_2 and $(w - w_0)$, we find

$$\begin{aligned}\tilde{\alpha}_1 &= \hat{g}_{22}X^1 - \hat{g}_{12}X^2 \\ \tilde{\alpha}_2 &= -\hat{g}_{12}X^1 + \hat{g}_{11}X^2 \\ (w - w_0) &= E\Theta^2 X^3 \quad \text{where} \\ \hat{g}_{j\kappa} &\equiv \frac{g_{j\kappa}}{g} \quad [\text{with } (j, \kappa) = (1, 2)] \quad ; \quad g \equiv g_{11}g_{22} - g_{12}^2 = \frac{1}{\hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}^2} = \frac{1}{\hat{g}}\end{aligned}\quad (3.13)$$

Hence, in terms of the thermodynamic forces, the electron and ion dimensionless entropy productions read, respectively

$$\begin{aligned}\Delta_I \hat{S}^e &= \frac{1}{2}E\Theta_e^2 X_e^{32} + \frac{1}{2}\hat{g}_{22}^e X_e^{12} + \frac{1}{2}\hat{g}_{11}^e X_e^{22} - \hat{g}_{12}^e X_e^1 X_e^2 + h.o.t. \\ \Delta_I \hat{S}^i &= \frac{1}{2}E\Theta_i^2 X_i^{22} + \frac{1}{2}\hat{g}_{11}^i X_i^{12} + h.o.t.\end{aligned}\quad (3.14)$$

After diagonalisation, the first expression of Eq. (3.14) can be brought into the form

$$\begin{aligned}\Delta_I \hat{S}^e &= \frac{1}{2}E\Theta_e^2 X_e^{32} + \nu_1 \xi_e^{12} + \nu_2 \xi_e^{22} \quad \text{with} \\ \nu_{1,2} &= \frac{1}{4} \left[(\hat{g}_{11} + \hat{g}_{22}) \pm \sqrt{(\hat{g}_{11} - \hat{g}_{22})^2 + 4\hat{g}_{12}^2} \right] \\ \xi_e^1 &= \frac{q_1}{h} \hat{g}_{12} X_e^1 - \frac{q_1}{2h} (\hat{g}_{11} - \hat{g}_{22} - h) X_e^2 \\ \xi_e^2 &= -\frac{q_2}{h} \hat{g}_{12} X_e^1 + \frac{q_2}{2h} (\hat{g}_{11} - \hat{g}_{22} + h) X_e^2 \\ h &= \sqrt{(\hat{g}_{11} - \hat{g}_{22})^2 + 4\hat{g}_{12}^2} \\ q_{1,2} &= \frac{\sqrt{4\hat{g}_{12}^2 + (\hat{g}_{11} - \hat{g}_{22})(\hat{g}_{11} - \hat{g}_{22} \pm h)}}{|\hat{g}_{12}| \sqrt{2}}\end{aligned}\quad (3.16)$$

Let us now consider the dimensionless electron entropy production estimated by the neoclassical theory for tokamak-plasmas in the weak-collisional transport regime [†]. Our aim is to find the values of the parameters such that the expression (3.14) coincides exactly with Eqs (3.7) *for all values of the thermodynamic forces*. Without considering the classical contributions, Eqs (3.7) may be rewritten as

$$\begin{aligned}\Delta_I \tilde{S}_{neocl}^e &= \tilde{\epsilon}_{\parallel}^e Z_e^{12} + \tilde{\kappa}_{\parallel}^e Z_e^{22} + \tilde{\sigma}_{\parallel} Z_e^{32} + 2\tilde{\delta}_{\parallel}^e Z_e^1 Z_e^2 + 2\tilde{\gamma}_{\parallel} Z_e^1 Z_e^3 + 2\tilde{\alpha}_{\parallel} Z_e^2 Z_e^3 \\ \Delta_I \tilde{S}_{neocl}^i &= \tilde{\epsilon}_{\parallel}^i Z_i^{12} + \tilde{\kappa}_{\parallel}^i Z_i^{22} + 2\tilde{\delta}_{\parallel}^i Z_i^1 z_i^2\end{aligned}\quad (3.17)$$

where

$$\left\{ \begin{array}{l} Z_e^1 = \bar{g}_{\parallel}^{e(5)} \\ Z_e^2 = g_{\parallel}^{e(3)} + \bar{g}_{\parallel}^{e(3)} \\ Z_e^3 \equiv g_{\parallel}^{(1)} - \bar{g}_{\parallel}^{e(1)} \end{array} \right. ; \quad \left\{ \begin{array}{l} Z_i^1 = g_{\parallel}^{i(3)} + \bar{g}_{\parallel}^{i(3)} \\ Z_i^2 = \bar{g}_{\parallel}^{i(5)} \end{array} \right. \quad (3.18)$$

As mentioned [see Eq. (3.2)], thermodynamic systems obtained by a transformation of forces and fluxes in such a way that the entropy production remains unaltered are thermodynamically equivalent [*Thermodynamic Covariance Principle* (TCP)] (Sonnino 2009), (G. Sonnino and A. Sonnino 2014). Hence, we should check whether the thermodynamic

[†] Here, calculations refer to electrons. For the ion entropy production the mathematical analysis is quite similar to the electron one. For ions, it is even simpler since the dimensionless ion entropy production is already diagonalized.

forces, X^μ and Z^μ , are linked each other by a linear transformation

$$\begin{cases} X_e^1 = Z_e^1 = \bar{g}_\parallel^{e(5)} \\ X_e^2 = Z_e^2 = g_\parallel^{e(3)} + \bar{g}_\parallel^{e(3)} \\ X_e^3 = a_e Z_e^1 + b_e Z_e^2 + c_e Z_e^3 \end{cases} \quad ; \quad \begin{cases} X_i^1 = Z_i^1 = g_\parallel^{i(3)} + \bar{g}_\parallel^{i(3)} \\ X_i^2 = a_i Z_i^1 + b_i Z_i^2 \end{cases} \quad (3.19)$$

These transformations leave invariant the entropy production i.e., $\Delta_I \hat{S}^\alpha = \Delta_I \hat{S}'^\alpha$ with $\alpha = (e, i)$ † and the two representation, $\{X^\mu\}$ and $\{Z^\mu\}$, are equivalent from the thermodynamic point of view (Prigogine 1954), (Sonnino 2009), (G. Sonnino and A. Sonnino 2014). Hence, by inserting Eqs (3.19) into Eq. (3.14), we get the expressions of dimensionless entropy production (per unit particle) in terms of the thermodynamic forces Z^μ . It is easily checked that the identities $\Delta_I \hat{S}'^\alpha \equiv \Delta_I \hat{S}_{neocl}^\alpha$ (with $\alpha = e, i$), are satisfied if, and only if,

$$\begin{aligned} a_e &= a \frac{2\tilde{\gamma}_\parallel}{\sqrt{E\tilde{\sigma}_\parallel}\Theta_e} \quad ; \quad b_e = \frac{2\tilde{\alpha}_\parallel}{\sqrt{E\tilde{\sigma}_\parallel}\Theta_e} \quad ; \quad c_e = \frac{2\sqrt{\tilde{\sigma}_\parallel}}{\sqrt{E}\Theta_e} \\ \hat{g}_{11}^e &= \frac{2}{\tilde{\sigma}_\parallel}(\tilde{\kappa}_\parallel^e \tilde{\sigma}_\parallel - \tilde{\alpha}_\parallel^2) \quad ; \quad \hat{g}_{22}^e = \frac{2}{\tilde{\sigma}_\parallel}(\tilde{\epsilon}_\parallel^e \tilde{\sigma}_\parallel - \tilde{\gamma}_\parallel^2) \quad ; \quad \hat{g}_{12}^e = \frac{2}{\tilde{\sigma}_\parallel}(\tilde{\alpha}_\parallel \tilde{\gamma}_\parallel - \tilde{\delta}_\parallel^e \tilde{\sigma}_\parallel) \quad (3.20) \\ a_i &= \frac{2\tilde{\delta}_\parallel^i}{\sqrt{E\tilde{\kappa}_\parallel^i}\Theta_i} \quad ; \quad b_i = \frac{2\sqrt{\tilde{\kappa}_\parallel^i}}{\sqrt{E}\Theta_i} \quad ; \quad \hat{g}_{11}^i = \frac{2}{\tilde{\kappa}_\parallel^i}(\tilde{\epsilon}_\parallel^i \tilde{\kappa}_\parallel^i - \tilde{\delta}_\parallel^i{}^2) \end{aligned}$$

To sum up, the set of Eqs (3.20) ensures that for low-collisional transport regime, the entropy production estimated by the neoclassical theory identifies with the exponent of the DDF Eq. (2.3). In the neoclassical theory, plasma is heated by ohmic power. Hence, the expression of $\Theta_\alpha(X)$ can be estimated by [see Eq. (3.4)]:

$$\int_{\mathcal{V}} d\mathbf{v} [\mathbf{v} - \mathbf{u}_\alpha(\mathbf{x}, \tilde{\Theta}_\alpha)] \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x}) \ln \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x}, \tilde{\Theta}_\alpha) = \frac{\mathbf{J}_{\mathcal{E}_{Oh.}}}{T_\alpha} \quad (3.21)$$

with $\mathbf{J}_{\mathcal{E}_{Oh.}}$ denoting the ohmic energy flux and [see Eqs (3.5, 3.6)]

$$\mathbf{u}_\alpha(\mathbf{x}, \Theta_\alpha) = \frac{\int_{\mathcal{V}} d\mathbf{v} \mathbf{v} \mathcal{F}^{\alpha R}(\Theta_\alpha, \mathbf{v}, \mathbf{x})}{\int_{\mathcal{V}} d\mathbf{v} \mathcal{F}^{\alpha R}(\Theta_\alpha, \mathbf{v}, \mathbf{x})} \quad ; \quad T_\alpha(\mathbf{x}) = \frac{1}{2} m_\alpha \frac{\int_{\mathcal{V}} d\mathbf{v} |\mathbf{v} - \mathbf{u}_\alpha|^2 \mathcal{F}^{\alpha R}(\mathbf{v}, \mathbf{x})}{\int_{\mathcal{V}} d\mathbf{v} \mathcal{F}^{\alpha R}(\Theta_\alpha, \mathbf{v}, \mathbf{x})} \quad (3.22)$$

In conclusion, the exponent of the DDF (2.3) coincides exactly with the entropy production estimated by the neoclassical theory (for tokamak-plasma in the weak-collisional regime) by setting [see Eqs (3.15, 3.20)]

$$\begin{aligned} \gamma &= 1 + E \quad ; \quad \Delta P_\phi^e = 1/\sqrt{\nu_1} \quad ; \quad \Delta \lambda_0^e \rightarrow \infty \quad ; \quad \Delta \lambda_1^e = 1/\nu_2 \\ \xi_e^1 &= P_\phi - P_{\phi 0} \quad ; \quad \xi_e^2 = \lambda - \lambda_0 \end{aligned} \quad (3.23)$$

Eqs (3.23) link the free parameters ΔP_ϕ^e , $\Delta \lambda_0^e$ and $\Delta \lambda_1^e$ with the transport coefficients. Note that in this case, $\Delta \lambda_0^e \rightarrow \infty$ and the presence of parameter $\Delta \lambda_1^e$ is crucial.

• The Local Equilibrium

At the local equilibrium, the \mathcal{F}_α^R , given in Eq. (2.3), reduces to a gamma distribution function. This happens when all thermodynamic forces vanish i.e., $X_e^1 = X_e^2 = 0$ and $X_i^1 = 0$. As an example of calculation, we considered L-mode, JET-plasma in the weak-

† Of course, the expressions of the flows, conjugate to the thermodynamic forces Z^μ , are modified in such a way that the dimensionless entropy production, $\Delta_I \hat{S}^\alpha$, remains invariant under the forces-transformation $\{X^\mu\} \rightarrow \{Z^\mu\}$.

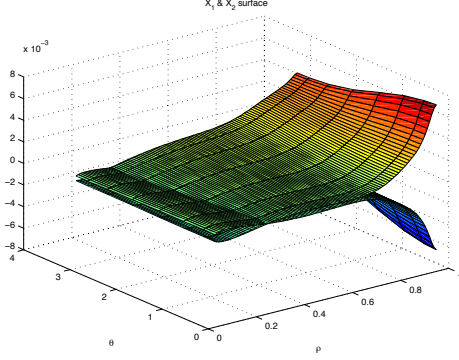


FIGURE 1. Examples of electron thermodynamic forces X_e^1 and X_e^2 . These electron thermodynamic forces have been obtained by using the profiles reported in Ref. (Mantica *et al.* 2003) for L-mode, JET-plasma. The intersection line corresponds to the values (ρ, θ) for which $X_e^1 = X_e^2 \simeq 0$ (up to the drift parameter to the square).

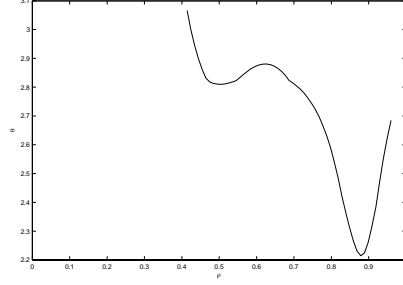


FIGURE 2. Example of the locus of the points (ρ, θ) where the ion thermodynamic force vanishes ($X_i^1 = 0$). The ion thermodynamic force has been obtained by using the profiles reported in Ref. (Mantica *et al.* 2003) for L-mode, JET-plasma.

collisional transport regime. By using the profiles reported in Ref. (Mantica *et al.* 2003), we estimated the electron and ion thermodynamic forces through the non-linear neoclassical theory (Sonnino 2009), (Sonnino and Peeters 2008). Fig. (1) illustrates the electron forces X_e^1 and X_e^2 . The intersection line corresponds to the locus of the points (ρ, θ) such that $X_e^1 = X_e^2 \simeq 0$ (with ρ and θ denoting the minor radius coordinate and the poloidal angle, respectively). Fig. (2) shows the curve $\rho = \rho(\theta)$ where the ion thermodynamic force, X_i^1 , vanishes.

The expressions (2.1) may be re-written as

$$P_\phi - \psi = \frac{B_0}{\Omega_{0c}} \frac{F v_\parallel}{|B|} \quad ; \quad \lambda - \frac{1}{2|B|} = -\frac{1}{2|B|} \frac{v_\parallel^2}{w} \quad (3.24)$$

Note that when $P_{\phi 0} \equiv \psi$ and $\lambda_0 \equiv 1/(2|B|)$, we have that the local equilibrium is reached for $v_\parallel = 0$. These conditions are useful when we analyze the weak-collisional transport regime and, in particular, the finite orbit width effects should be considered. In this case in fact, the term $(B_0 F v_\parallel)/(\Omega_{0c} |B|)$ corresponds to the deviation of the orbit of the particle from the poloidal flux surface $P_\phi = \psi$. At $P_\phi = \psi$ we have $v_\parallel = 0$ i.e., the whole parallel kinetic energy of the particle is converted into perpendicular energy, and the particle is reflected, as in a mirror.

4. Conclusions

We have shown that the expression of a stationary distribution function, recently proposed in literature (Di Troia 2012) and derived by non-equilibrium statistical mechanics (Sonnino *et al.* 2013)-(Sonnino *et al.* 2014), can be re-obtained solely by the information theory *i.e.*, by extremizing the Shannon functional, subject to *ad hoc* scale-invariant restrictions (MaxEnt principle). From the physical point of view, the special choice of the scale invariance restrictions can be justified by evoking the theorem of the thermodynamic of irreversible processes, such as the Minimum Entropy Production principle (MEP).

For concreteness, we analyzed the simplest case of fully ionized, L-mode, Tokamak-plasmas, in the weak collisional transport regime. We showed that it is possible to set the expression of the free parameters of the DDF obtained by MaxEnt principle, in such a way that the logarithm of the DDF identifies exactly with entropy production estimated by the neoclassical theory. In other words, the stationary DDF can be understood under the framework of Prigogine's probability theory for thermodynamic fluctuations. These results lead to the suggestive idea that the stationary DDFs used for heating Tokamak-plasmas by NBI (Neutral Beam Injection) or by ICRH (Ion Cyclotron Radio Heating) may also be re-obtained by the information theory. However, as shown in Ref. (Sonnino and Steinbrecher 2014), Eq. (2.3) is unable to treat dynamical systems in highly anisotropic phase space such as turbulent systems or anisotropic dynamical systems subject to random perturbations. Ref. (Sonnino and Steinbrecher 2014) suggests that these cases may equally be treated by the information theory. The price to pay is that the MaxEnt principle should be applied to more sophisticated expressions of entropy functionals, such as the generalized Rényi entropies (GRE). These entropies play the role of Liapunov functionals.

This series of works opens also another interesting perspective. Through the thermodynamical field theory (TFT) (Sonnino 2009), it is possible to estimate the DDF when the nonlinear contributions cannot be neglected (Sonnino 2011). The task should be then to establish the link between the stationary DDFs, derived by the MaxEnt principle applied to GRE, subject to scale-invariant restrictions, with the ones found by the TFT. All of this will be subject of forthcoming works.

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